Ratios of B and D Meson Decay Constants in Relativistic Quark Model

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Abstract

We calculate the ratios of B and D meson decay constants by applying the variational method to the relativistic hamiltonian of the heavy meson. We adopt the Gaussian and hydrogen-type trial wave functions, and use six different potentials of the potential model. We obtain reliable results for the ratios, which are similar for different trial wave functions and different potentials. The obtained ratios show the deviation from the nonrelativistic scaling law, and they are in a pretty good agreement with the results of the Lattice calculations.

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The knowledge of the decay constant of the B meson f_B is very important, since it affects the magnitude of $B - \bar{B}$ mixing and the size of CP violation significantly. There have been intensive theoretical and experimental researches for improving its understanding. However, its theoretical calculation is difficult because it is in the realm of nonperturbative QCD and the motion of the light quark in B meson is relativistic. Understanding the decay constant better is also invaluable because its information can reveal the inside structure of the hadron.

Grinstein [1] observed that the double ratio of the decay constants $(f_{B_s}/f_{B_d})/(f_{D_s}/f_{D_d})$ is very close to 1 with small correction of order m_s/m_Q. He calculated the double ratio with the heavy quark effective theory, and obtained 0.967. He remarked that the value of f_{B_s}/f_{B_d}, which is an important factor for the relative strengths of $B_s - \bar{B}_s$ and $B_d - \bar{B}_d$ mixings, can then be given reliably from the knowledge of the measurable f_{D_s}/f_{D_d}. Oakes [2] also calculated the double ratio based on the assumption that chiral symmetry is broken by quark mass terms in the Lagrangian. He obtained its value as 1.004, and emphasized the importance of the fact that this double ratio is very close to 1.

When one treats the heavy-light meson in analogy with the nonrelativistic situation, one expects the scaling law $f_B/f_D \simeq \sqrt{M_D/M_B}$, since the reduced masses of the light quark (u or d quark) in B and D mesons have almost the same value, and $f_P^2 M_P = 12\,|\psi(0)|^2$ by the Van Royen-Weisskopf formula [3] for the pseudoscalar meson P, where $\psi(0)$ is the wave function at origin of the relative motion of quarks [4]. However, the light quark inside B or D meson has large velocity, and its nonrelativistic treatment is not legitimate. Indeed, our calculation of the decay constants in the relativistic quark model, which we present in this Letter, shows that the nonrelativistic consideration is much deviated by the relativistic motion of the light quark, since this relativistic nature makes the $\psi(0)$ of B and D mesons different appreciably. This character of the relativistic motion has also been exposed by the Lattice calculations [5], since they have obtained larger values

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of f_B/f_D than $\sqrt{M_D/M_B}$ from the nonrelativistic scaling law. This situation can be understood clearly through our relativistic calculation.

The potential model has been successful for ψ and Υ families with the nonrelativistic hamiltonian, since their heavy quarks can be treated nonrelativistically. However, for D or B meson it has been difficult to apply the potential model because of the relativistic motion of the light quark in D or B meson. In our calculation we work with the purely relativistic hamiltonian, and adopt the variational method [6]. We take the Gaussian and hydrogen-type wave functions separately as trial wave functions [7], and obtain the ground state energy and wave function by minimizing the expectation value of the relativistic hamiltonian. By using the wave function at origin $\psi(0)$, we can obtain the value of the decay constant f_P from the Van Royen-Weisskopf formula. Then we can obtain the ratios of the decay constants. The reason why we choose the Gaussian and hydrogen-type trial wave functions is that the former one is appropriate to the long range confining potential, and the latter one to the short range asymptotically free potential.

The heavy-light pseudoscalar meson is composed of one heavy quark with mass m_Q and one light quark with m_q, and its relativistic hamiltonian is given by

$$H = \sqrt{\mathbf{p}^2 + m_Q{}^2} + \sqrt{\mathbf{p}^2 + m_q{}^2} + V(r), \tag{1}$$

where $\mathbf{r}$ and $\mathbf{p}$ are the relative coordinate and its conjugate momentum. The hamiltonian in (1) represents the energy of the meson in the center of mass coordinate, since in that reference frame the momenta of both the heavy and light quarks have the same magnitude as that of the conjugate momentum of the relative coordinate.

We apply the variational method to the hamiltonian (1) with the Gaussian and hydrogen-type trial wave functions. The Gaussian wave function is given by

$$\psi(\mathbf{r}) = (\frac{\mu}{\sqrt{\pi}})^{3/2} e^{-\mu^2 r^2/2}, \tag{2}$$

where μ is the variational parameter. The Fourier transform of $\psi(\mathbf{r})$ gives the

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momentum space wave function $\chi(\mathbf{p})$, which is also Gaussian,

$$\chi(\mathbf{p}) = \frac{1}{(\sqrt{\pi}\mu)^{3/2}} e^{-p^2/2\mu^2}. \tag{3}$$

The ground state is given by minimizing the expectation value of H in (1),

$$\langle H \rangle = \langle \psi | H | \psi \rangle = E(\mu), \quad \frac{d}{d\mu} E(\mu) = 0 \quad \text{at} \quad \mu = \bar{\mu}, \tag{4}$$

and then $\bar{\mu} \equiv p_F$ represents the inverse size of the meson, and $\bar{E} \equiv E(\bar{\mu})$ its mass M_P [6]. For the value of the light quark mass m_q in (1), we use the current quark mass given by Dominguez and Rafael [8]: $m_d = 9.9$ MeV and $m_s = 199$ MeV.

We perform the same calculation as the above for the hydrogen-type wave function

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{4\pi}} \frac{2}{a_0^{3/2}} e^{-r/a_0}, \tag{5}$$

where a_0 is the variational parameter which represents the size of the meson. The momentum space wave function conjugate to the $\psi(\mathbf{r})$ in (5) is given by

$$\chi(\mathbf{p}) = \frac{2\sqrt{2}}{\pi} \frac{a_0^{3/2}}{(a_0^2 p^2 + 1)^2}. \tag{6}$$

For $V(r)$ in (1), we consider the following six potentials of the potential model, which we also display in Fig. 1. We note in Fig. 1 the tendency that the potential which has higher values of potential energy in the short range has lower values in the long range, and vice versa.

(A) Coulomb and linear potential of Eichten et $al.$ [9]:

$$V(r) = -\frac{\alpha_c}{r} + Kr, \tag{7}$$

with $\alpha_c = 0.52$, $K = 1/(2.34)^2$ GeV^2, $m_c = 1.84$ GeV, $m_b = 5.18$ GeV.

(B) Coulomb and linear potential of Hagiwara et $al.$ [10]:

$$V(r) = -\frac{\alpha_c}{r} + Kr, \tag{8}$$

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with $\alpha_c = 0.47$, $K = 0.19$ GeV^2, $m_c = 1.32$ GeV, $m_b = 4.75$ GeV.

(C) Power law potential of Martin [11]:

$$V(r) = -8.064 \text{ GeV} + (6.898 \text{ GeV}) (r \cdot 1 \text{ GeV})^{0.1}, \qquad (9)$$

with $m_c = 1.8$ GeV, $m_b = 5.174$ GeV.

(D) Power law potential of Rosner $et\ al.$ [12]:

$$V(r) = -0.772 \text{ GeV} + 0.801 \left((r \cdot 1 \text{ GeV})^\alpha - 1 \right) / \alpha, \qquad (10)$$

with $\alpha = -0.12$, $m_c = 1.56$ GeV, $m_b = 4.96$ GeV.

(E) Logarithmic potential of Quigg and Rosner [13]:

$$V(r) = -0.6635 \text{ GeV} + (0.733 \text{ GeV}) \log(r \cdot 1 \text{ GeV}), \qquad (11)$$

with $m_c = 1.5$ GeV, $m_b = 4.906$ GeV.

(F) Richardson potential [14]:

$$V(r) = \frac{8\pi}{33 - 2n_f} \Lambda (\Lambda r - \frac{f(\Lambda r)}{\Lambda r}), \ f(t) = 1 - 4 \int_1^\infty \frac{dq}{q} \frac{e^{-qt}}{[\ln(q^2 - 1)]^2 + \pi^2}, \qquad (12)$$

with $n_f = 3$, $\Lambda = 0.398$ GeV, $m_c = 1.491$ GeV, $m_b = 4.884$ GeV.

The results of the variational calculations with the Gaussian wave function are organized in Table 1, and those with the hydrogen-type wave function in Table 2. We see in Table 1 and 2 that the larger energy ($\bar{E}$) state has the smaller size of meson ($1/\bar{\mu}$ or a_0). In order to check whether the Gaussian wave function in (2) is a really good wave function, we enlarged the trial wave function by adding the second excited harmonic oscillator eigenfunction which is an even function, since the first excited one which is an odd function can not be included in the ground state wave function of the relativistic hamiltonian which commutes with the parity operator. For this enlarged trial wave function we obtained the result that the Gaussian part contributes much dominantly, therefore it confirms that the Gaussian wave

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function is a very good trial wave function in the variational calculation of the relativistic hamiltonian in (1) with the harmonic type wave function. We also checked the wave function in (5) by adding the first excited hydrogen-type wave function, and confirmed that the wave function in (5) is also much dominant for the ground state wave function of (1).

The decay constant f_P of the pseudoscalar meson P is defined by the matrix element $\langle 0|A_\mu|P(q)\rangle$:

$$\langle 0|A_\mu|P(q)\rangle = iq_\mu f_P. \tag{13}$$

By considering the low energy limit of the heavy meson annihilation, we have the relation between f_P and the ground state wave function at origin $\psi_P(0)$ from the Van Royen-Weisskopf formula with the color factor [3]:

$$f_P^2 = \frac{12}{M_P}|\psi_P(0)|^2, \tag{14}$$

where M_P is the heavy meson mass. Using this formula, from (2) and (5) we have

$$
\begin{aligned}
f_P &= \sqrt{\frac{12}{M_P}} \left(\frac{p_F(P)}{\sqrt{\pi}} \right)^{3/2} && \text{for Gaussian wave function,} & (15)\\
&= \sqrt{\frac{12}{M_P}} \left(\frac{1}{\pi^{1/3}a_0(P)} \right)^{3/2} && \text{for hydrogen} - \text{type wave function.} & (16)
\end{aligned}
$$

Then the ratio of the decay constants of pseudoscalar mesons A and B is given by [6]

$$
\begin{aligned}
\frac{f_A}{f_B} &= \sqrt{\frac{M_B}{M_A}} \times \left(\frac{p_F(A)}{p_F(B)} \right)^{3/2} && \text{for Gaussian wave function,} & (17)\\
&= \sqrt{\frac{M_B}{M_A}} \times \left(\frac{a_0(B)}{a_0(A)} \right)^{3/2} && \text{for hydrogen} - \text{type wave function.} & (18)
\end{aligned}
$$

By using the values in Table 1 and 2 for M ($\bar{E}$), p_F ($\bar{\mu}$), and a_0 in (17) and (18), we obtain the ratios of the decay constants for the potential models (A)-(F), which we present in Table 3 and 4. We see in Table 3 and 4 that f_{B_s}/f_{D_s} is enhanced, compared with the nonrelativistic scaling law $\sqrt{M_{D_s}/M_{B_s}}$ whose experimental value is 0.605, by the factor of 1.324 which is induced by the factor

of $|\psi_{B_s}(0)/\psi_{D_s}(0)|$. For the estimation of this enhancement factor we used the average of the fourth columns of Table 3 and 4. Sometimes this factor has been approximated to be 1 and $f_{B_s}/f_{D_s} \simeq \sqrt{M_{D_s}/M_{B_s}}$ has been used, by treating it in analogy with the nonrelativistic case [4]. However our calculation shows that this factor is indeed important and different from 1 significantly. The same situation happens for the ratio f_{B_d}/f_{D_d}. Compared with $\sqrt{M_{D_d}/M_{B_d}}$ whose experimental value is 0.595, f_{B_d}/f_{D_d} is enhanced by the factor of 1.335, which is estimated from the average of the fifth columns of Table 3 and 4. This situation is in agreement with the results of the Lattice calculations [5], as we see in Table 5, where we organized the results of Lattice calculations.

As an application of the obtained ratios of decay constants, let us consider the determination of the values of f_{D_d}, f_{B_s}, and f_{B_d} from the experimental value of f_{D_s}. The cleanest way to obtain the value of the decay constant from the experimental results is through the purely leptonic decays of the D_s^+ meson, which are understood theoretically to occur via an annihilation of the two valence quarks. The decay rate of the D_s^+ meson is given by the formula [19]

$$\Gamma(D_s^+ \to l^+\nu) \;=\; \frac{1}{8\pi}\, G_F^2\, f_{D_s}^2\, m_l^2\, M_{D_s} \left(1 - \frac{m_l^2}{M_{D_s}^2}\right)^2 |V_{cs}|^2, \qquad (19)$$

where f_{D_s} is the meson decay constant, M_{D_s} is the D_s mass, m_l is the mass of the final-state lepton, G_F is the Fermi coupling constant, and V_{cs} is the CKM matrix element. The WA75 and CLEO collaborations took the data for the branching ratio $B(D_s^+ \to \mu^+\nu)$ [20, 21], and the Review of Particle Properties [22] presents $B(D_s^+ \to \mu^+\nu) = (5.9 \pm 2.2) \times 10^{-3}$. By using this branching ratio, the life time of D_s meson $\tau = (0.467 \pm 0.017) \times 10^{-12}$ s, $M_{D_s} = 1968.5 \pm 0.7$ MeV, $m_\mu = 105.66$ MeV, $G_F = 1.1664 \times 10^{-5}$ GeV^{-2}, and $|V_{cs}| = 1.01 \pm 0.18$ [22], we obtain the following value of f_{D_s} from (19):

$$f_{D_s} \;=\; 265 \pm 68 \text{ MeV}. \qquad (20)$$

The uncertainty of the value of f_{D_s} is due to those of the experimental values of

$B(D_s^+ \to l^+\nu)$ and $|V_{cs}|$, therefore if their values are improved experimentally, we can obtain f_{D_s} very accurately. When we combine the f_{D_s} value in (20) and the ratios given by the average of those in Table 3 and 4, we get the following values of f_{D_d}, f_{B_s}, and f_{B_d}: $f_{D_d} = 253 \pm 65$ MeV, $f_{B_s} = 212 \pm 54$ MeV, $f_{B_d} = 201 \pm 51$ MeV.

In summary, we calculated the various ratios of the B and D meson decay constants by applying the variational method to the relativistic hamiltonian. We took the Gaussian and hydrogen-type trial wave functions separately, and used six different potentials for the potential energy. We obtained the results which are similar for different trial wave functions and different potentials. This fact implies that our method is reliable. The obtained results for the ratio f_B/f_D show that the nonrelativistic scaling law $f_B/f_D \simeq \sqrt{M_D/M_B}$ should be implemented by the relativistic consideration. Its enhancement factor we obtained is about 1.33, which is induced by $|\psi_B(0)/\psi_D(0)|$. This result is in a pretty good agreement with the recent Lattice calculations. Our results for f_{B_s}/f_{B_d} and f_{D_s}/f_{D_d} are both about 1.05. We also obtained the value of the double ratio $(f_{B_s}/f_{B_d})/(f_{D_s}/f_{D_d})$ as 1.008, whereas Grinstein obtained 0.967 with the heavy quark effective theory, and Oakes 1.004 with the chiral symmetry breaking.

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Model	$\bar{\mu}(B_s)$	$\bar{E}(B_s)$	$\bar{\mu}(B_d)$	$\bar{E}(B_d)$	$\bar{\mu}(D_s)$	$\bar{E}(D_s)$	$\bar{\mu}(D_d)$	$\bar{E}(D_d)$
A (Eich.)	0.565	5.933	0.544	5.896	0.495	2.661	0.478	2.620
B (Hagi.)	0.550	5.553	0.530	5.516	0.466	2.217	0.451	2.174
C (Power 1)	0.587	5.154	0.567	5.119	0.510	1.855	0.493	1.815
D (Power 2)	0.600	5.353	0.577	5.318	0.495	2.042	0.475	2.000
E (Log.)	0.586	5.396	0.565	5.360	0.490	2.080	0.472	2.038
F (Rich.)	0.584	5.404	0.564	5.368	0.495	2.103	0.479	2.062
(Average)	0.579	5.466	0.558	5.430	0.492	2.160	0.475	2.118

Table 1: The values of the variational parameter μ which minimize $\langle H \rangle$, and the corresponding values of the minimum energy for the Gaussian wave function

Model	$a_0(B_s)$	$\bar{E}(B_s)$	$a_0(B_d)$	$\bar{E}(B_d)$	$a_0(D_s)$	$\bar{E}(D_s)$	$a_0(D_d)$	$\bar{E}(D_d)$
A (Eich.)	1.260	5.884	1.310	5.845	1.549	2.140	1.605	2.094
B (Hagi.)	1.309	5.512	1.360	5.472	1.570	2.189	1.625	2.143
C (Power 1)	1.305	5.118	1.353	5.078	1.504	1.823	1.561	1.779
D (Power 2)	1.264	5.303	1.317	5.264	1.532	1.997	1.602	1.951
E (Log.)	1.308	5.200	1.360	5.160	1.593	1.876	1.660	1.829
F (Rich.)	1.260	5.359	1.307	5.321	1.503	2.068	1.558	2.024
(Average)	1.284	5.396	1.335	5.357	1.542	2.016	1.602	1.970

Table 2: The values of the variational parameter a_0 which minimize $\langle H \rangle$, and the corresponding values of the minimum energy for the hydrogen-type wave function.

Model	f_{B_s}/f_{B_d}	f_{D_s}/f_{D_d}	$\dfrac{(f_{B_s}/f_{B_d})}{(f_{D_s}/f_{D_d})}$	f_{D_s}/f_{B_s}	f_{D_d}/f_{B_d}
A (Eich.)	1.055	1.046	1.009	1.225	1.236
B (Hagi.)	1.054	1.040	1.013	1.234	1.250
C (Power 1)	1.050	1.041	1.009	1.350	1.362
D (Power 2)	1.057	1.053	1.004	1.213	1.218
E (Log.)	1.053	1.047	1.005	1.232	1.238
F (Rich.)	1.050	1.040	1.010	1.251	1.263
(Average)	1.053±.003	1.045±.005	1.008±.003	1.251±.046	1.261±.047

Table 3: Ratios of the decay constants obtained for the Gaussian wave function.

Model	f_{B_s}/f_{B_d}	f_{D_s}/f_{D_d}	$\dfrac{(f_{B_s}/f_{B_d})}{(f_{D_s}/f_{D_d})}$	f_{D_s}/f_{B_s}	f_{D_d}/f_{B_d}
A (Eich.)	1.057	1.043	1.013	1.217	1.232
B (Hagi.)	1.055	1.042	1.013	1.208	1.223
C (Power 1)	1.052	1.044	1.007	1.354	1.363
D (Power 2)	1.060	1.057	1.003	1.221	1.224
E (Log.)	1.056	1.050	1.006	1.239	1.246
F (Rich.)	1.053	1.044	1.008	1.236	1.246
(Average)	1.056±.003	1.047±.005	1.008±.004	1.246±.050	1.256±.049

Table 4: Ratios of the decay constants obtained for the hydrogen-type wave function.

Group	f_{B_s}/f_{B_d}	f_{D_s}/f_{D_d}	$\frac{(f_{B_s}/f_{B_d})}{(f_{D_s}/f_{D_d})}$	f_{D_s}/f_{B_s}	f_{D_d}/f_{B_d}
ELC [15]	$1.08\pm.06$	$1.08\pm.02$	$1.00\pm.06$	$1.03\pm.22$	$1.02\pm.21$
UKQCD[16]	$1.22\,^{+.04}_{-.03}$	$1.18\pm.02$	$1.03\,^{+.04}_{-.03}$	$1.09\,^{+.04+.42}_{-.03-.06}$	$1.16\,^{+.05+.46}_{-.05-.14}$
BLS [17]	$1.11\pm.02\pm.05$	$1.11\pm.02\pm.05$	$1.00\pm.03\pm.06$	$1.11\pm.06\pm.27$	$1.11\pm.08\pm.30$
MILC [18]	$1.13(2)(9)(4)$	$1.09(1)(4)(4)$	$1.04(2)(9)(5)$	$1.18(3)(17)(13)$	$1.22(5)(17)(19)$

Table 5: The results of the ratios from Lattice calculations: the third column was estimated from the values of the first and second columns, and the fourth and fifth columns were estimated from the Lattice calculation results of f_{D_s}, f_{B_s}, f_{D_d}, and f_{B_d}.

Fig. 1. The potentials (A)-(E) given in Eqs. (7)-(12). The radial distance of the

horizontal axis is in the unit of GeV^{-1} ($1\ \mathrm{GeV}^{-1} = 0.197$ fm), and the potential energy of the vertical axis is in the unit of GeV.